Matroid Intersection

Problem Given matroids $M_1=(S,\mathcal{J}_1)$ and $M_2=(S,\mathcal{J}_2)$

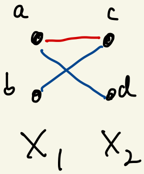
Find $I\in\mathcal{J}_1\cap\mathcal{J}_2$ such that $|I|\ge|I'|\ \forall I'\in\mathcal{J}_1\cap\mathcal{J}_2$

Recall that $X=(S,\mathcal{J}_1\cap\mathcal{J}_2)$ is not always a matroid

For example, let $B=(X_1,X_2,E)$ be a bipartite graph and

let $\mathcal{J}_i=\{E'\subseteq E \mid d_{E'}(v)\le 1\ \forall v\in X_i\}$ $i=1,2$

Then $M_i=(E,\mathcal{J}_i)$ is a matroid but $\mathcal{J}_1\cap\mathcal{J}_2$ does not satisfy the exchange axiom:

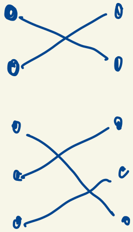


$I'=\{ac\}$ $I=\{bc,ad\}$

$|I|>|I'|$ but cannot add any edge from I to I

But we can solve maximum bipartite matching as a matroid intersection problem for the matroids M_1, M_2 above:

matchings

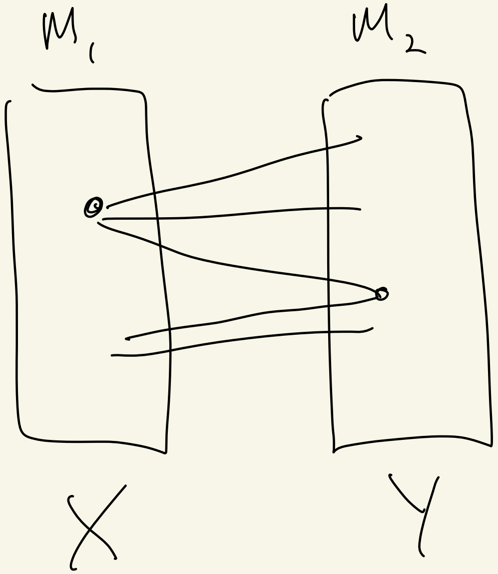


independent in both M_1, M_2

so $\max\{|M| \mid M \text{ is a matching}\}$

$= \max\{|I| \mid I\in\mathcal{J}_1\cap\mathcal{J}_2\}$

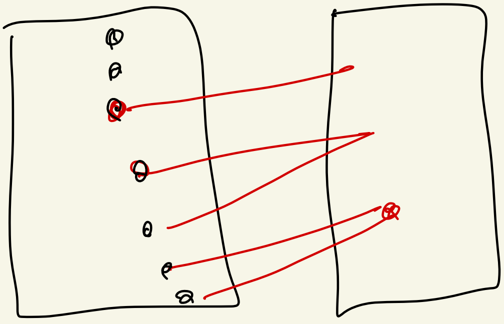
$B=(X,Y,E)$



in M_1 a set $E' \subseteq E$ is independent

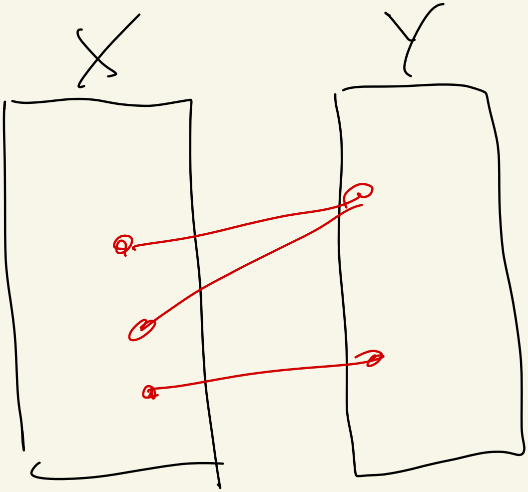
if $d_{E'}(v) \le 1 \quad \forall v \in X$

--- M_2 $d_{E'}(u) \le 1 \quad \forall u \in Y$

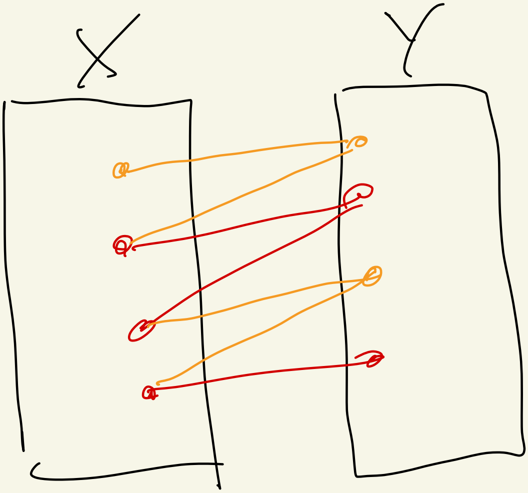


independent in M_1
but not in M_2

Why is M_i a matroid?



(2) holds



— independent in M_1

— ind. in M_1

Another example of matroid intersection: out-branchings

Given $D=(V,A)$ and $s\in V$ define

$$\mathcal{I}_1=\{A' \mid A' \text{ is a forest in } UG(D)\}$$

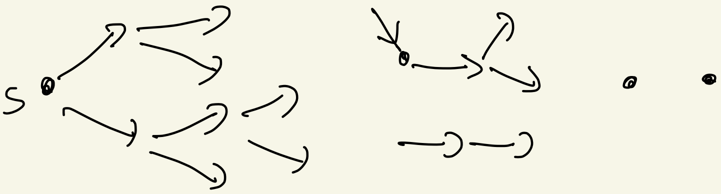
UG(D) is the underlying undirected graph of D

$$\mathcal{I}_2=\{A'' \mid d^-_{A''}(v)\le 1 \ \forall v\in V-s \text{ and } d^-_{A''}(s)=0\}$$

- We have seen that $M_1=(A,\mathcal{I}_1)$ is a matroid as it is the circuit matroid of UG(D)
- $M_2=(A,\mathcal{I}_2)$ is a matroid as $\mathcal{I}_2$ satisfies the three axioms $\emptyset\in\mathcal{I}_2$, $X\subseteq\mathcal{I}_2 \wedge Y\subseteq X \Rightarrow Y\in\mathcal{I}_2$, $X,Y\in\mathcal{I}_2, |Y|=|X|+1 \Rightarrow \exists y\in Y\setminus X$ s.t. $X+y\in\mathcal{I}_2$

Now $\max\{|I| \mid I\in\mathcal{I}_1\cap\mathcal{I}_2\}=n-1$

$\Updownarrow$ D has an out-branching from s



independent sets in $\mathcal{I}_1\cap\mathcal{I}_2$

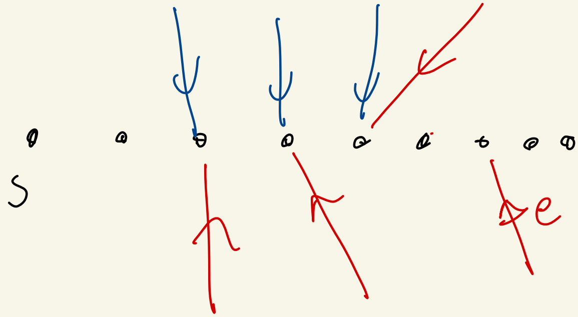
No cycles in underlying graph + indegree ≤ 1 and $d^-(s)=0$

M_2

X blue

Y red

$|Y| \geq |X|+1$



$X+e \in \mathcal{I}_2$

Recall that if $M=(S,\mathcal{J})$ is a matroid and $X\in\mathcal{J}$ but $X+e\notin\mathcal{J}$, then is a unique circuit (minimal dependent set) C in $X+e$

For any $X\in\mathcal{J}$ and $e\in S$ let

- $C(X,e)=\emptyset$ if $X+e\in\mathcal{J}$
- $C(X,e)$ = unique circuit in $X+e$ if $X+e\notin\mathcal{J}$

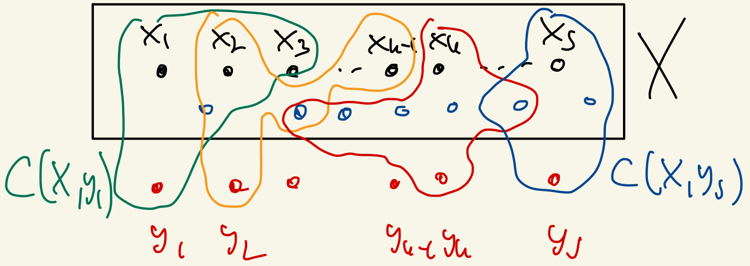
Below we follow Korte & Vygen Section 13.5

Lemma 13.27 Let $M=(E,\mathcal{J})$ be a matroid and let $X\in\mathcal{J}$ and suppose that $x_1,x_2,\dots,x_s$ are distinct elements of X and $y_1,y_2,\dots y_s$ are distinct elements of $E-X$ such that

(a) $x_k\in C(X,y_k)$ for $k=1,2,\dots,s$

(b) $x_j\notin C(X,y_k)$ for all j with $1\le j<k\le s$

Then $X-\{x_1,x_2,\dots,x_s\}+\{y_1,y_2\dots y_s\}\in\mathcal{J}$



Proof: we show by induction on r that $X_r \in \mathcal{J}$

when $X_r = X - \{x_1, x_2, \dots, x_r\} + \{y_1, \dots, y_r\}$

$r = 0$ ok as $X_0 = X$

suppose $X_{r-1} \in \mathcal{J}$

if $X_{r-1} + y_r \in \mathcal{J}$ then $X_r = (X_{r-1} + y_r) - x_r \in \mathcal{J}$

otherwise $\exists$ unique circuit C in $X_{r-1} + y_r$

By (b) none of $x_1, x_2, \dots, x_{r-1}$ belongs to $C(X, y_r)$

so $C(X, y_r) \subseteq X_{r-1} + y_r$ implying $C = C(X, y_r)$

By (a) $x_r \in C(X, y_r)$ so $X_{r-1} + y_r - x_r \in \mathcal{J}$ $\square$.

Edmonds matroid intersection algorithm:

- Start with $X = \emptyset$
- Augment X by one element at a time until no new element can be added

How to do this?

Given current set $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ define a digraph $G_X = (E, A_X^{(1)} \cup A_X^{(2)})$

Here $A_X^{(1)} = \{x \to y \mid y \in E - X \wedge x \in C_1(X,y) - y\}$ C_1: circuit of M_1

$A_X^{(2)} = \{y \to x \mid y \in E - X \wedge x \in C_2(X,y) - y\}$ C_2: circuit of M_2

This means that

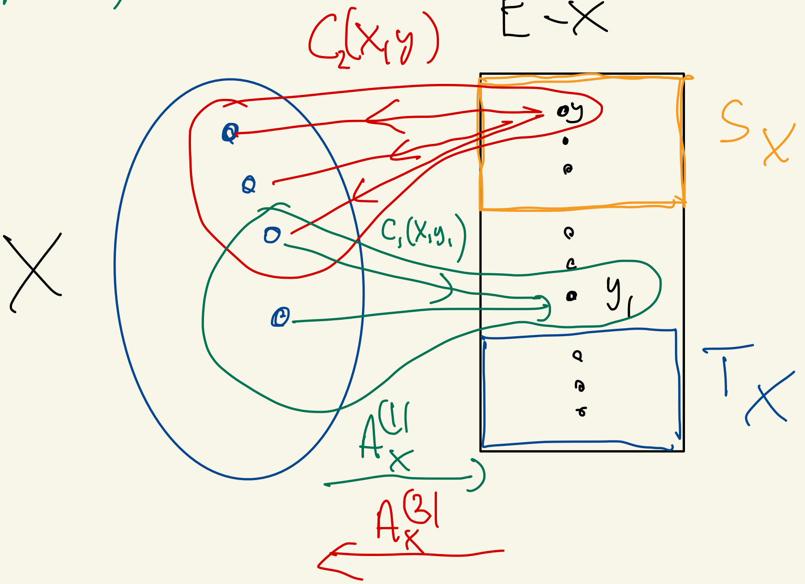
In $A_X^{(1)}$ an arc $x \to y$ indicates that by deleting x we may add y to X so that $X + y - x \in \mathcal{I}_1$

In $A_X^{(2)}$ an arc $y \to x$ indicates that by deleting x we may add y to X so that $X + y - x \in \mathcal{I}_2$

Let $S_X = \{y \in E - X \mid X + y \in \mathcal{I}_1\}$, $T_X = \{y \in E - X \mid X + y \in \mathcal{I}_2\}$

If $S_X = \emptyset$ or $T_X = \emptyset$ then X is $\underline{\text{maximal}}$ in $\mathcal{I}_1 \cap \mathcal{I}_2$

If $y \in S_X \cap T_X$ then $X + y \in \mathcal{I}_1 \cap \mathcal{I}_2$ so we can add y directly to X. So assume $S_X \cap T_X = \emptyset$



Lemma 13.28. Suppose $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ and that

$$P = y_0 \rightarrow x_1 \rightarrow y_1 \rightarrow x_2 \rightarrow y_2 \rightarrow \cdots \rightarrow x_s \rightarrow y_s$$

is a shortest (S_X, T_X)-path in $G_X = (E, A_X^{(1)} \cup A_X^2)$

Then $X' = (X + \{y_0, y_1, \ldots, y_s\}) - \{x_1, x_2, \ldots x_s\}$ is in $\mathcal{I}_1 \cap \mathcal{I}_2$ $(\square)$

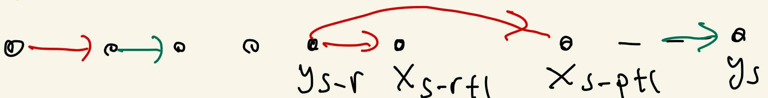
(note that $|X'| = |X| + 1$)

Proof We show that $X + y_0$, $x_1, x_2, \ldots x_s$ and $y_1, y_2 \ldots y_s$ satisfy the conditions of Lemma 13.27 wrt M_1. This will imply that $X' \in \mathcal{I}_1$

- $X + y_0 \in \mathcal{I}_1$ as $y_0 \in S_X$
- (a) holds as $x_j \rightarrow y_j \in A_X^{(1)}$ for $j = 1, 2 \ldots s$
- To see that (b) holds, assume that $x_j \in C_1(X, y_k)$ for some $1 \leq j < k \leq s$. Then $x_j \rightarrow y_k \in A_X^1$ implying that P is <u>not</u> a shortest (S_X, T_X)-path, contradiction

Similarly we can show that $X' \in \mathcal{I}_2$ by showing that $X + y_s$, $x'_1, x'_2, \ldots x'_s$ and $y'_1, y'_2, \ldots, y'_s$ with $x'_i = x_{s-i+1}$ $y'_i = y_{s-i}$ satisfy the conditions of Lemma 13.27 wrt M_2.

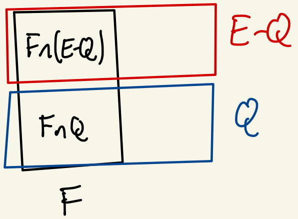
- $X + y_s \in \mathcal{I}_2$ as $y_s \in T_X$
- (a) holds as $y_j \rightarrow x_{j+1} \in A_X^2$ for $j = 0, 1, \ldots, s-1$
- To see that (b) holds, assume that $x'_p \in C_2(X, y'_r)$ for some $p < r$ then the arc $y_{s-r} \rightarrow x_{s-p+1}$ contradicts that P is shortest



Proposition 13.29 Let $M_i=(E,\mathcal{J}_i)$ $i=1,2$ be matroids and let r_i, $i=1,2$ be their rank functions. Then

$\forall F\in \mathcal{J}_1\cap\mathcal{J}_2$ and $\forall Q\subseteq E$: $|F|\le r_1(Q)+r_2(E-Q)$

Proof:



$F\cap Q\in\mathcal{J}_1$ so $|F\cap Q|\le r_1(Q)$

$F\cap(E-Q)\in\mathcal{J}_2$ so $|F\cap(E-Q)|\le r_2(E-Q)$

$\Downarrow$

$|F|=|F\cap Q|+|F\cap(E-Q)|\le r_1(Q)+r_2(E-Q)$

□.

Lemma 13.30 $X\in\mathcal{J}_1\cap\mathcal{J}_2$ has maximum size

$\Updownarrow$

G_X has no (S_X,T_X)-path

Proof

$\Downarrow$ follows from Lemma 13.28

$\Uparrow$: Let $R=\{e\in E \mid \exists$ path from S_X to e in $G_X\}$

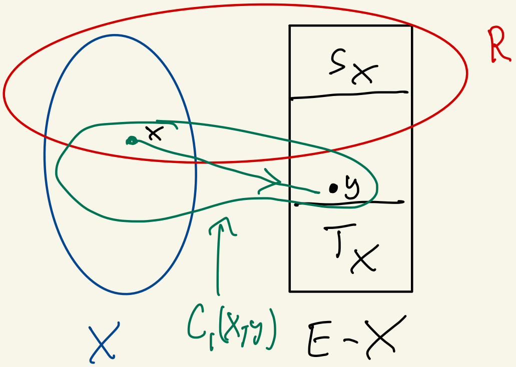
Then $S_X\subseteq R$ and $R\cap T_X=\emptyset$

We claim that $r_1(X-R)=|X-R|$ and $r_2(R)=|X\cap R|$

If this holds we can take $Q=E-R$ and get

$$|X|=|X-R|+|X\cap R|=r_1(X-R)+r_2(R)$$
$$=r_1(Q)+r_2(E-Q)$$
so $|X|$ is maximum by Prop 13.29

- Suppose $r_1(E-R) > |X-R|$ then $\exists y \in (E-R)-X$ such that $(X-R)+y \in \mathcal{J}_1$



But $X+y \notin \mathcal{J}_1$ as $y \notin S_X \subseteq R$

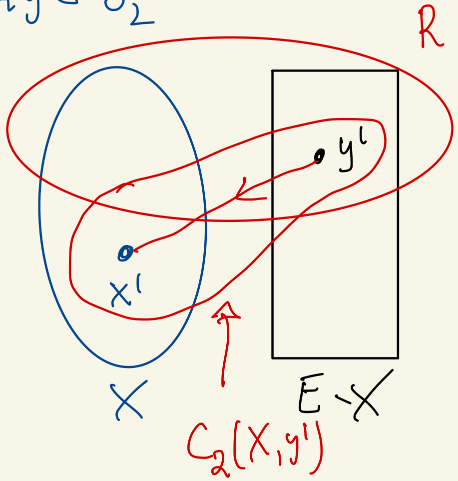
$(y \in (E-R)-X \subseteq E-X)$

Now $C_1(X,y) \cap (X \cap R) \neq \emptyset$ as

$(X-R)+y \in \mathcal{J}_1$.

But then there is an arc $x \to y$ in A'_X from R to y implying that $y \in R$ contradiction (as $y \in (E-R)-X$)

- Suppose $r_2(R) > |X \cap R|$
  Then $\exists y' \in R-X$ s.t. $X \cap R + y' \in \mathcal{J}_2$



But $X+y' \notin \mathcal{J}_2$ as $R \cap T_X = \emptyset$

Then $C_2(X,y') \cap X-R \neq \emptyset$

as $(X \cap R)+y' \in \mathcal{J}_2$

Now there is an arc $y' \to x'$

from R to E-R ↯

We have shown that

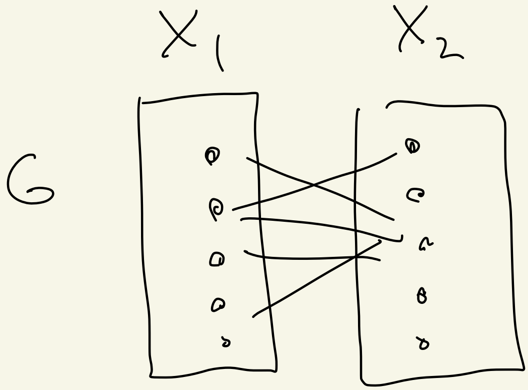
$$r_1(E-R) = |X-R| \text{ and } r_2(R) = |X \cap R|$$

Theorem (Edmonds matroid intersection theorem)

Let $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ be matroids over E. Then $X \in \mathcal{I}_1 \cap \mathcal{I}_2$ is of maximum size if and only if $|X| = \min_{Q \subseteq E} r_1(Q) + r_2(E-Q)$

Application to bipartite matchings

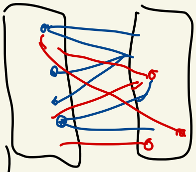
$M_i := (E, J_i)$



$J_i = \{E' \subseteq E \mid d_{E'}(v) \leq 1 \; \forall v \in X_i\}$

Size of maximum $X \in J_1 \cap J_2$ = maximum size of a matching in G

By Edmonds thm



$$\boxed{1} \quad \max |X| = \min_{E' \subseteq E} \{ r_1(\underline{E'}) + r_2(\underline{E - E'}) \}$$

note $r_i(E^*)$ = # vertices in X_i covered by E^*

so

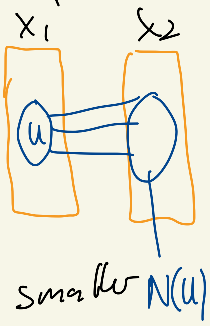
$$\max |X| = \min \{ |U_1| + |U_2| \mid U_i \subseteq X_i \wedge U_1 \cup U_2 \text{ is a vertex cover} \}$$

König's thm

Deriving Hall's theorem:

Thm (Hall) $G=(X_1,X_2,E)$ with $|X_1|=|X_2|$ has a perfect matching if and only if

$\forall U \subseteq X_1 \quad |N(U)| \geq |U| \qquad (\square)$

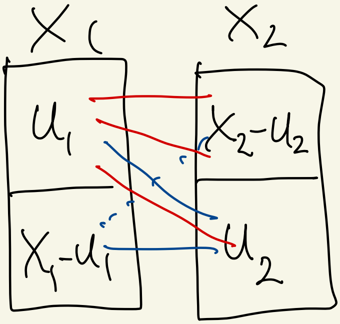


P: necessity is clear

suppose that size of max matching is smaller than $|X_1|$. Then by Edmonds thm $\exists E' \subseteq E$ such that

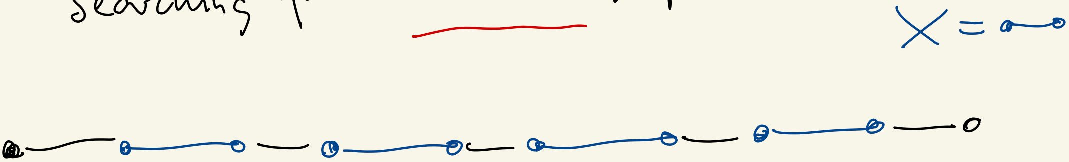
$|X_1| > r_1(\underline{E'}) + r_2(\underline{E-E'}) = |U_1| + |U_2|$ when $U_1 \subseteq X_1$ covers E' and $U_2 \subseteq X_2$ covers $E-E'$

Then $|U_2| < |X| - |U_1|$ but all neighbours of $X_1 - U_1$ are in U_2 $\lightning$ $(\square)$



Observation

Edmonds algorithm corresponds to searching for alternating paths:

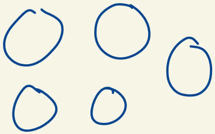


Thm (Berge)

a matching M in a graph $G = (V, E)$ is maximum if and only if G has no alternating path wrt M

Rainbow spanning trees

Theorem (Suzuki–Schrijver) An edge coloured graph $G=(V,E)$ has a rainbow (all colours different) spanning tree



$\Updownarrow$ $\forall$ partitions $V_1, V_2 \dots V_k$, $k \geq 2$ of V

There are at least $k-1$ different colours between $V_1, V_2 \dots V_k$

P: clearly necessary as every spanning tree needs at least $k-1$ edges to connect $V_1, V_2, \dots V_k$

Define $M_1=(E,\mathcal{J}_1)$ and $M_2=(E,\mathcal{J}_2)$ by

$\mathcal{J}_1=\{E'\subseteq E \mid \text{all edges in } E' \text{ have different colours}\}$

$\mathcal{J}_2=\{E'\subseteq E \mid E' \text{ is a forrest (acyclic)}\}$

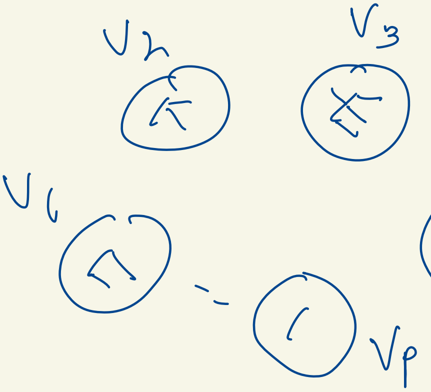
M_1 and M_2 are matroids:

By Edmonds theorem G has a rainbow spanning tree if and only if

$$(*)\quad \min_{E'\subseteq E}\left\{ r_1(E') + r_2(E-E') \right\} \geq n-1$$

note that $r_2(E-E') = n - \#$components in $G(E-E')$

denote by p



If colour(e) = colour(e^*) with $e^* \in E-E'$ and $e \in E'$ then move e^* to E'

this will only lower $(*)$

$r_1(E')$ = no of colours on edges in E'

By $(*)$ $\quad r_1(E') \geq (n-1)-(n-p) \geq p-1$